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Wormhole solutions in Einstein–Yang–Mills–Higgs system: II. Zero-order structure for $\mathcal{G} = \mathrm{SU}(2)$†

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Abstract. For a simple Einstein–Yang–Mills–Higgs model, we study properties of the extreme wormhole solutions. As shown in a recent paper, the values of all fields at the boundary of the hole must satisfy a system of differential equations. We find solutions to this system in a number of cases, among others a non-abelian family similar to the 't Hooft–Polyakov monopole. We also prove that all solutions must be axisymmetric, and conjecture that the set of solutions which we have found is already complete.

1. Introduction

In a recent paper (Hajicek 1982, hereafter denoted by I), we have derived a system of differential equations which must be satisfied by the boundary values of the metric, Yang–Mills potentials and Higgs scalar at the internal infinity of a wormhole. We have assumed that the wormhole is a static and smooth solution to an Einstein–Yang–Mills–Higgs system of a very general type: we admitted an arbitrary gauge group $\mathcal{G}$, an arbitrary representation for the Higgs field and an arbitrary Higgs potential. For a broad class of potentials, as well as for each Higgs vacuum of any potential, we have shown that all solutions must be abelian.

The purpose of the present paper is to study the properties of solutions to the equations mentioned above, if the potential does not belong to the 'abelian' class. In particular, we want to show that there are then non-abelian solutions. We restrict ourselves to the most simple model possible to simplify the equations.

The model we choose is defined as follows. The gauge group $\mathcal{G}$ is SU(2), the Higgs field Q is in the adjoint representation of $\mathcal{G}$, the quadratic forms introduced in I are given by

$$(\ ,\ )_g = -(2e^2)^{-1}(\ ,\ )^K, \qquad (\ ,\ )_q = -\tfrac{1}{2}(\ ,\ )^K,$$

and the potential function $V(Q)$ reads

$$V(Q) = \tfrac{1}{8}k[(Q, Q)_q - F^2]^2,$$

where k and F are positive constants (with $k = 0$ or $F = 0$, $V(Q)$ belongs to the 'abelian' class of theorem 2 in I).

We choose the following basis for the Lie algebra su(2) of SU(2):

$$(T^a)_{bc} = \epsilon^{abc}.$$

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Then

$$[T^a, T^b] = (T^a)_{cb} T^c = -\epsilon^{abc} T_c,$$

and, for any X, $Y \in \mathrm{su}(2)$, we have: if $X = X^a T^a$ and $Y^a T^a$, then

$$(X, Y)^K = -2X^a Y^a.$$

One can therefore write

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \epsilon^{abc} W^b_\mu W^c_\nu, \qquad (D_\mu Q)^a = \partial_\mu Q^a + \epsilon^{abc} W^b_\mu Q^c,$$

$$(\partial V/\partial Q)^a = \tfrac{1}{2} k (Q^b Q^b - F^2) Q^a, \qquad \omega^a(Q_1, Q_2) = e^2 \epsilon^{abc} Q^b_1 Q^c_2.$$

Then, the complete system of equations for the boundary values of the fields (the so-called 0-order structure), as given by equations (48)–(54) in I, can be written in the following form:

$$1/G^2 = \gamma^2 (E^a E^a/e^2 + B^a B^a/e^2 - 2V - \Lambda/\gamma^2), \tag{1}$$

$$R = 2\gamma^2 (E^a E^a/e^2 + B^a B^a/e^2 + g^{AB} (D_A Q)^a (D_B Q)^a + 2V + \Lambda/\gamma^2), \tag{2}$$

$$E^a E^a/e^2 + B^a B^a/e^2 - 2V = \text{constant}, \tag{3}$$

$$(D_A Q)^a (D_B Q)^a = \tfrac{1}{2} g_{AB} g^{KL} (D_K Q)^a (D_L Q)^a, \tag{4}$$

$$(D_A E)^a = 0, \qquad \epsilon^{abc} E^b Q^c = 0, \tag{5}$$

$$(\epsilon^{AB}/\sqrt{g})(D_B B)^a + e^2 g^{AB} \epsilon^{abc} Q^b (D_B Q)^c = 0, \tag{6}$$

$$(1/\sqrt{g})(D_A(\sqrt{g} g^{AB} D_B Q))^a - \tfrac{1}{2} k (Q^b Q^b - F^2) Q^a = 0. \tag{7}$$

Here, the notation is taken over from I. The only difference is that we leave out the symbol '$\lim_{N \to 0}$' and understand tacitly that all fields take on their boundary values.

The plan of the paper is as follows. In § 2, we solve the equations (1)–(7) assuming that the electric field E^a does not everywhere vanish. It seems to be a particular property of just SU(2) that there are only abelian solutions in this case. Indeed, already the group SU(3) looks more promising, because it contains for example the subgroup U(1) × SU(2). Equations (5) can then be satisfied by taking a generator of U(1) for E, all other fields being u(1) × su(2)-algebra valued so that they commute with E. This suggests that more-dimensional groups will lead to a larger number of different soliton solutions. We shall study this question elsewhere.

In § 3, we describe the promised non-abelian solution. It can exist only if the parameters in the Lagrangian satisfy certain conditions, in particular, the 'vacuum' value of the Higgs field is of the order of the Planck mass. The solution is spherically symmetric and satisfies the 't Hooft ansatz. If there is a reasonable global solution with this boundary value, then it can be something like the gravitating 't Hooft–Polyakov monopole with an extreme black hole in the middle. We also show that all spherically symmetric solutions to the equations (1)–(7) are exhausted by the spherical abelian solutions given by theorem 2 in I and our non-abelian solution.

In § 4, we prove that any analytic solution to equations (1)–(7) must be axially symmetric. Extended calculations which we performed in studying the system (1)–(7) for axially symmetric fields on topologically spherical manifolds (but which we are not going to publish here) suggest the following conjecture.

Conjecture. All solutions to the system (1)–(7) consist of the abelian solutions given by theorem 2 in I and the non-abelian solution given in § 3 of the present paper.

2. The case of non-vanishing electric field

In this section, we assume that

$$E^a E^a \neq 0.$$

Then the whole system (1)–(7) is easily solvable. Let us choose a gauge on H such that

$$E^a = \mathscr{E} n^a,$$

where $\mathscr{E}$ is a function in H and

$$n^a = (1, 0, 0).$$

Then the longitudinal part of the first equation (5) gives

$$\mathscr{E} = \text{constant}, \tag{8}$$

whereas the transversal part is equivalent to

$$W^a_A = W_A n^a \tag{9}$$

with W_A being a covariant vector field in H. The second equation (5) means that

$$Q^a = q n^a \tag{10}$$

where q is a function in H.

Equation (9) and the definition of B (see I, equation (17)) imply

$$B^a = \mathscr{B} n^a, \tag{11}$$

where

$$\mathscr{B} = (1/\sqrt{g})(\partial_2 W_3 - \partial_3 W_2). \tag{12}$$

Now, equation (6) is equivalent to

$$\mathscr{B} = \text{constant}, \tag{13}$$

whereas (7) becomes

$$(1/\sqrt{g})\partial_A(\sqrt{g} g^{AB} \partial_B q) - \tfrac{1}{2}k(q^2 - F^2)q = 0. \tag{14}$$

Equation (14) is the only non-trivial remainder from the Yang–Mills–Higgs part of the system (1)–(7). We can simplify it further using the Einstein equation (3): with (8) and (13), it yields

$$q = \text{constant}. \tag{15}$$

Then (4) is satisfied and (14) is equivalent to

$$(q^2 - F^2)q = 0.$$

Thus, we have one of the two possible Higgs vacuums; this leads immediately to the solutions described in I. We must only set in theorem 2

$$\bar{e}_2 = 0, \qquad X = X_1, \qquad w = 0$$

and set

$$2\gamma^2 V = 0, \qquad q = F,$$

for the true vacuum and

$$2\gamma^2 V = \tfrac{1}{4}k\gamma^2 F^4, \qquad q = 0$$

for the false one.

3. The case of vanishing electric field: the non-abelian solution

In this section, we assume that

$$E^a E^a = 0.$$

By that, equations (5) are both satisfied and we have no restriction on W^a_A and Q^a like (9) and (10).

Of course, there will be the solutions of theorem 2 in I: if $Q^aQ^a = F^2$ and $Q^aQ^a = 0$. Next, we show that any solution must satisfy

$$0 \leq Q^aQ^a \leq F^2 \tag{16}$$

on the whole of H. For assume that

$$Q^aQ^a > F^2 \tag{17}$$

at some point, p say, of H. Then there is an open subset H_0 of H such that (17) holds in H_0 and $Q^aQ^a = F^2$ at ∂H_0 (∂H_0 can be empty). Clearly, we have at ∂H_0

$$\nu^A \partial_A (Q^aQ^a) \leq 0,$$

where ν^A is an external unit normal vector to ∂H_0. If we now multiply equation (7) by $Q^a\sqrt{g}$ and integrate over H_0, we obtain

$$-\int_{H_0} d^2x\, \sqrt{g}[\tfrac{1}{2}k(Q^aQ^a - F^2)Q^bQ^b + g^{AB}(D_AQ)^a(D_BQ)^a]$$
$$+\oint_{\partial H_0} ds\, \nu^A Q^a (D_AQ)^a = 0. \tag{18}$$

However,

$$\nu^A Q^a (D_AQ)^a = \nu^A \tfrac{1}{2}\partial_A(Q^aQ^a) \leq 0$$

and (18) can hold only if

$$\partial_A(Q^aQ^a) = 0$$

at ∂H_0 and

$$Q^aQ^a = F^2, \qquad (D_AQ)^a = 0,$$

on H_0; this is the desired contradiction.

Any new solution has therefore the property that it satisfies (16) and $Q^aQ^a = 0$, F^2 only at isolated points. Let us look for such solutions.

We choose the gauge on H such that

$$Q^a = qn^a, \qquad n^a = (1, 0, 0), \tag{19}$$

and split W^a_A into longitudinal and transversal components

$$W^a_A = W_A n^a + U^a_A, \qquad U^a_A = (\delta^{ab} - n^a n^b) W^b_A. \tag{20}$$

The remaining gauge freedom is represented by the transformations

$$W'_A = W_A + \chi_A,$$
$$U'^2_A = U^2_A \cos\chi + U^3_A \sin\chi, \qquad U'^3_A = -U^2_A \sin\chi + U^3_A \cos\chi; \tag{21}$$

thus, we have an 'electromagnetic' potential W_A, and a 'charged vector' field, $U_A = U^2_A + \mathrm{i}U^3_A$, on the manifold H.

Using the gauge (19) and the splitting (20), we have

$$(D_AQ)^a(D_BQ)^a = (\partial_A q)(\partial_B q) + q^2 U^a_A U^a_B, \qquad \epsilon^{abc}Q^b(D_AQ)^c = q^2 U^a_A.$$

The Einstein equations (1)–(4) become

$$1/G^2 = \gamma^2(B^aB^a/e^2 - 2V(q) - \Lambda/\gamma^2), \tag{22}$$
$$R^2 = 2\gamma^2(B^aB^a/e^2 + g^{KL}(\partial_K q)(\partial_L q) + q^2U^2 + 2V + \Lambda/\gamma^2), \tag{23}$$
$$B^aB^a/e^2 - 2V(q) = \text{constant}, \tag{24}$$
$$(\partial_A q)(\partial_B q) + q^2 U^a_A U^a_B = \tfrac{1}{2} g_{AB}(g^{KL}(\partial_K q)(\partial_L q) + q^2U^2), \tag{25}$$

where

$$U^2 = g^{KL}U^a_K U^a_L$$

is a gauge- and coordinate-invariant quantity. Equation (6) yields

$$(D_AB)^a = e^2q^2\sqrt{g}\epsilon_{ABg}{}^{BC}U^a_c. \tag{26}$$

Written out, this reads as follows:

$$\partial_A B^1 + U^2_A B^3 - U^3_A B^2 = 0, \tag{27}$$
$$\partial_A B^2 + U^3_A B^1 - W_A B^3 = e^2q^2\sqrt{g}\epsilon_{ABg}{}^{BC}U^2_C, \tag{28}$$
$$\partial_A B^3 + W_A B^2 - U^2_A B^1 = e^2q^2\sqrt{q}\epsilon_{ABg}{}^{BC}U^3_c. \tag{29}$$

The definition of the magnetic field (see I, equation (17)) is

$$B^1 = (1/\sqrt{g})(\partial_2 W_3 - \partial_3 W_2 + U^2_2 U^3_3 - U^3_2 U^2_3), \tag{30}$$
$$B^2 = (1/\sqrt{g})(\partial_2 U^3_3 - \partial_3 U^3_2 + W_3 U^2_2 - W_2 U^3_3), \tag{31}$$
$$B^3 = (1/\sqrt{g})(\partial_2 U^3_3 - \partial_3 U^3_2 + W_2 U^2_3 - W_3 U^2_2). \tag{32}$$

Finally, the scalar equation (7) reads:

$$(1/\sqrt{g})\partial_A(\sqrt{g}g^{AB}\partial_B q) - \tfrac{1}{2}qU^2 + \tfrac{1}{2}k(F^2 - q^2)q = 0, \tag{33}$$
$$q(1/\sqrt{g})\partial_A(\sqrt{g}g^{AB}U^a_B) + g^{AB}(U^a_B\partial_A q + U^a_A\partial_B q) - qg^{AB}W_A\epsilon^{ab}U^b_B = 0, \tag{34}$$

where ϵ^{ab} is the usual antisymmetric symbol in two dimensions, $a, b = 2, 3$.

We are not able to solve the nonlinear elliptic system of equations (22)–(25) and (27)–(33) completely. We can, however, find the desired non-abelian solution, and show that any solution must be axisymmetric (next section).

Theorem 1. Assume that $q = \text{constant}$, $0 < q < F$, on H. Then, the system (22)–(25) and (27)–(33) has a regular solution only if

$$(\gamma F)^2 > \tfrac{1}{2} \tag{35}$$

and, distinguishing the following three cases, if

$$4\gamma^2\Lambda/k = \tfrac{1}{4} - [(\gamma F)^2 - \tfrac{1}{2}]^2, \tag{36}$$

$$\tfrac{1}{2} \leq (\gamma q)^2 < (\gamma F)^2, \tag{37}$$

for $k = 4e^2$ (case (i)),

$$\frac{e^2}{k} - [(\gamma F)^2 - \tfrac{1}{2}]^2 \leq \frac{4\gamma^2\Lambda}{k} < \frac{e^2}{k} - \frac{4e^2}{k}[(\gamma F)^2 - \tfrac{1}{2}]^2, \tag{38}$$

$$(\gamma q)^2 = \frac{1}{2} + \left[\frac{k}{4e^2 - k}\left(\frac{e^2}{k} - [(\gamma F)^2 - \tfrac{1}{2}]^2 - \frac{4\gamma^2\Lambda}{k}\right)\right]^{1/2}, \tag{39}$$

for $k < 4e^2$ (case (ii)), and

$$\frac{e^2}{k} - \frac{4e^2}{k}[(\gamma F)^2 - \tfrac{1}{2}]^2 < \frac{4\gamma^2\Lambda}{k} \leq \frac{e^2}{k} - [(\gamma F)^2 - \tfrac{1}{2}]^2, \tag{40}$$

$$(\gamma q)^2 = \frac{1}{2} + \left[\frac{k}{k - 4e^2}\left(-\frac{e^2}{k} + [(\gamma F)^2 - \tfrac{1}{2}]^2 + \frac{4\gamma^2\Lambda}{k}\right)\right]^{1/2}, \tag{41}$$

for $k > 4e^2$ (case (iii)). The solution is then spherically symmetric with radius r_0, and in a suitable gauge and the spherical coordinates θ, φ, is given by

$$(\gamma/r_0)^2 = \tfrac{1}{4}k(\gamma F)^4 + \tfrac{1}{4}(4e^2 - k)(\gamma q)^4 + \gamma^2\Lambda, \tag{42}$$

$$(\gamma/G)^2 = e^2(\gamma q)^4 - \tfrac{1}{4}k[(\gamma F)^2 - (\gamma q)^2]^2 - \gamma^2\Lambda, \tag{43}$$

$$W_2 = U_3^2 = U_2^3 = 0, \tag{44}$$

$$W_3 = \pm\cos\theta, \qquad U_2^2 = \tfrac{1}{2}r_0[k(F^2 - q^2)]^{1/2}, \qquad U_3^3 = \pm U_2^2 \sin\theta. \tag{45}$$

Comment. Figures 1, 2 and 3 illustrate the inequalities (38), (36) and (40).

In case (i), the solution depends on one free parameter, namely the value of q within the limits (37). The lower limit corresponds to $G = \infty$, the upper one to the abelian solution with $q = F$. As q increases, the scalar energy density at the horizon,

$$(e^2/2\gamma^4)[(\gamma F)^2 - (\gamma q)^2]^2,$$

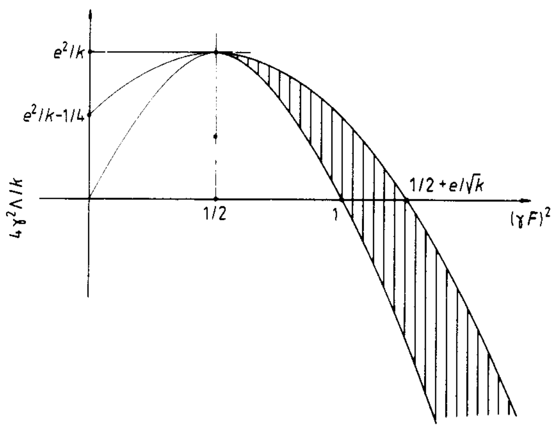

Figure 1. For $k < 4e^2$, the non-abelian solution exists only if the coupling constants lie in the shaded region.

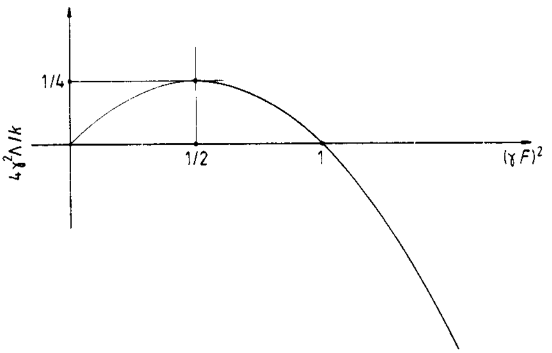

Figure 2. For $k = 4e^2$, the non-abelian solution exists only if the coupling constants lie in the curve indicated.

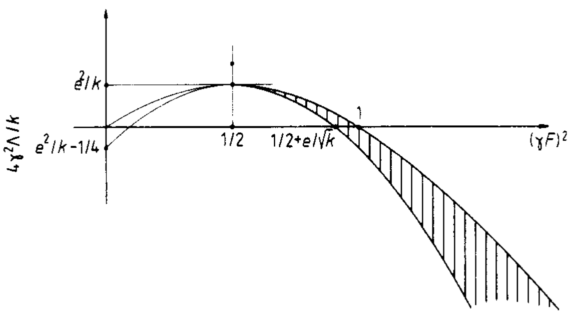

Figure 3. For $k > 4e^2$, the non-abelian solution exists only if the coupling constants lie in the shaded region.

decreases from $(e^2/2\gamma^4)[(\gamma F)^2 - \frac{1}{4}]^2$ to zero, whereas the Yang–Mills energy density,

$$(e^2/2\gamma^4)(\gamma q)^4,$$

increases from $e^2/8\gamma^4$ to $(e^2/2\gamma^4)(\gamma F)^4$. Their sum is not constant; the geometry changes through the quantity G, the radius r_0 of the hole does not depend on $(\gamma q)^2$.

In cases (ii) and (iii), the solution does not contain any free parameter—it is completely determined by the values of the coupling constants γ, e, k, F, Λ. In case (ii), the lower limit (38) for $4\gamma^2\Lambda/k$ corresponds to the abelian solution with $q = F$, the upper one to $G = \infty$. In case (iii), exactly the opposite is true for (40).

One can change the gauge in such a way that the transformed Yang–Mills potentials and the scalar at H satisfy the 't Hooft ansatz (van Nieuwenhuizen *et al* 1976). Clearly, we must rotate first around the 1-axis in the internal space, separately on the north and south hemisphere, to regularise the field at the poles, and then, again separately, one has to rotate Q^a into the 'radial' direction. The calculation is straightforward, so we skip it.

Proof of theorem 1. We can always choose the gauge such that $B^2 = 0$. Let us multiply equation (26) by B^a; we obtain

$$\partial_A(B^aB^a) = 2e^2q^2B^3\sqrt{q}\epsilon_{AB}g^{BC}U_c^3. \tag{46}$$

Taking gradients of both sides of (24), we have from (46)

$$\partial_A q = -(2/k)[q^2/(F^2-q^2)]B^3\sqrt{g}\epsilon_{AB}g^{BC}U^3_C. \tag{47}$$

We show easily that

$$B^3 = 0. \tag{48}$$

Assume, to the contrary, that $B^3 \neq 0$. Then, (47) implies $U^3_A = 0, A = 2, 3$. Setting this in equation (25), we obtain

$$U^2_A U^2_B = \tfrac{1}{2}g_{AB}U^2.$$

This equation can be satisfied with a non-degenerate metric g_{AB} only, if $U^2_A = 0, A = 2, 3$. However, equation (32) implies, for $U^2_A = U^3_A = 0, B^3 = 0$, and this is the desired contradiction.

From $q =$ constant and equations (33), (24) and (22), (23), it follows that (H, g_{AB}) is a two-dimensional space of constant curvature. Indeed, (33) and (24) yield

$$B^a B^a = \text{constant}$$

and

$$U^2 = \tfrac{1}{2}k(F^2-q^2) = \text{constant},$$

so that from (22)

$$1/G^2 = \text{constant}$$

and from (23)

$$R = 2\gamma^2[\tfrac{1}{2}kq^2(F^2-q^2) + B^aB^a/e^2 + 2V + \Lambda/\gamma^2] = \text{constant}. \tag{49}$$

If $R > 0$, the topology is spherical and we have a spherically symmetric geometry; if $R = 0$, the topology is toroidal and the geometry is locally plane symmetric; $R =$ constant < 0 cannot be realised on a compact manifold H. Consider first the spherical case. Choose the spherical coordinates θ, φ with

$$ds^2 = r_0^2\, d\theta^2 + r_0^2 \sin^2\theta\, d\varphi^2,$$

and rotate U^a_2 by a gauge transformation (21) so that

$$U^3_2 = 0, \qquad U^2_2 > 0. \tag{50}$$

(The gauge condition $B^2 = 0$ is empty because of (48).) Then (25) reads

$$U^2_2 U^2_3 = 0, \qquad (U^2_2)^2 = \tfrac{1}{4}kr_0^2(F^2-q^2),$$
$$(U^2_3)^2 + (U^3_3)^2 = \tfrac{1}{4}kr_0^2 \sin^2\theta\, (F^2-q^2).$$

This together with (50) is equivalent to the last two equations of (44) and (45).

Now, (31) and (32) imply the first equations of (44) and (45). With (44) and (45), equations (33) and (34) are identically satisfied, and (28) and (29) yield

$$B^1 = \mp e^2 q^2. \tag{51}$$

Now, (27) is identically satisfied, whereas (30) is equivalent to

$$(\gamma/r_0)^2 = \tfrac{1}{4}[k(\gamma F)^2 + (4e^2-k)(\gamma q)^2]. \tag{52}$$

From (48) and (51),

$$B^aB^a/e^2 = e^2q^4,$$

and the remaining equations, namely (22) and (23), imply (43) and (42). Equations (42) and (52) are compatible only if

$$(4e^2-k)(\gamma q)^4-(4e^2-k)(\gamma q)^2+k[(\gamma F)^4-(\gamma F)^2]+4\gamma^2\Lambda=0. \qquad (53)$$

The roots of this equation must satisfy the following additional conditions:

$$0<(\gamma q)^2<(\gamma F)^2 \qquad (54)$$

(assumption of theorem 1) and

$$(4e^2-k)(\gamma q)^4+2k(\gamma F)^2(\gamma q)^2-k(\gamma F)^4-4\gamma^2\Lambda \geqslant 0 \qquad (55)$$

(reality of G and (43)). The last inequality can be brought to a more convenient form by adding the equality (53) to it. This yields

$$[2(\gamma q)^2-1]\{4e^2(\gamma q)^2+k[(\gamma F)^2-(\gamma q)^2]\}\geqslant 0.$$

The second factor is always non-negative because of (54), so we have, equivalently to (55):

$$(\gamma q)^2 \geqslant \tfrac{1}{2}, \qquad (56)$$

and the scalar field is always of the order of the Planck mass.

We consider the following sub-cases: (i) $k=4e^2$, (ii) $k<4e^2$, (iii) $k>4e^2$.

(i) $k=4e^2$

Equation (53) leads to

$$(\gamma F)^2_\pm = \tfrac{1}{2}\pm(\tfrac{1}{4}-4\gamma^2\Lambda/k)^{1/2}. \qquad (57)$$

The inequalities (54) and (56) exclude $(\gamma F)^2_-$. In order that $(\gamma F)^2_+$ is positive, we must have

$$4\gamma^2\Lambda/k \leqslant \tfrac{1}{4}.$$

However, the equality would allow no solution to (54) and (56).

Thus, in order to have a solution, Λ is uniquely given by means of γ, F, and k: solving equation (57) yields (36), and F must satisfy (35). q is arbitrary within the limits (54) and (56); using (57) we obtain (37).

(ii) $k<4e^2$

From the roots of equation (53), the inequality (56) leaves only (39). The reality of $(\gamma q)^2$ is equivalent to

$$4\gamma^2\Lambda/k \leqslant e^2/k-[(\gamma F)^2-\tfrac{1}{2}]^2, \qquad (58)$$

and (54) yields (using the fact that $(\gamma F)^2>\tfrac{1}{2}$)

$$4\gamma^2\Lambda/k > e^2/k-(4e^2/k)[(\gamma F)^2-\tfrac{1}{2}]^2. \qquad (59)$$

As $4e^2/k>1$ in this case, the inequalities (52) and (53) have a solution, except for $(\gamma F)^2=\tfrac{1}{2}$, and are equivalent to (38).

(iii) $k>4e^2$

Again we have only (41), but instead of (58) and (59), we obtain

$$4\gamma^2\Lambda/k \geqslant e^2/k-[(\gamma F)^2-\tfrac{1}{2}]^2 \qquad (60)$$

and

$$4\gamma^2\Lambda/k < e^2/k - (4e^2/k)[(\gamma F)^2 - \tfrac{1}{2}]^2. \tag{61}$$

(60) and (61) are compatible except for $(\gamma F)^2 = \frac{1}{2}$, and are equivalent to (40).

Next, consider $R = 0$, the toroidal topology and the locally plane symmetric geometry. In the most general case, the torus has the metric

$$ds^2 = r_1^2\, d\theta^2 + r_2^2\, d\varphi^2 - 2r_1r_2 \cos\alpha\, d\theta\, d\varphi, \tag{62}$$

where

$$0 \leqslant \theta \leqslant 2\pi, \qquad 0 \leqslant \varphi \leqslant 2\pi,$$

are periodical coordinates and r_1, r_2, α are parameters,

$$r_1 > 0, \qquad r_2 > 0, \qquad 0 < \alpha < \pi.$$

Using the gauge (50) again, we arrive at

$$U_2^2 = \tfrac{1}{2}r_1[k(F^2 - q^2)]^{1/2}, \qquad U_2^3 = 0, \tag{63}$$

$$U_3^2 = -r_2(\cos\alpha)\tfrac{1}{2}[k(F^2 - q^2)]^{1/2}, \tag{64}$$

$$U_3^3 = \pm r_2(\sin\alpha)\tfrac{1}{2}[k(F^2 - q^2)]^{1/2}. \tag{65}$$

Equations (31) and (32) now imply

$$W_2 = W_3 = 0. \tag{66}$$

With (62)–(66), equations (33) and (34) are identically satisfied, whereas (28) and (29) are equivalent to

$$B^1 = \mp e^2q^2.$$

Then (27) is identically satisfied, but (30) becomes

$$e^2q^2 = -\{\tfrac{1}{2}[k(F^2 - q^2)]\}^2,$$

which is impossible. Thus, there are no toroidal solutions.

4. The case of vanishing electric field: axial symmetry

In this section, we show the following theorem.

Theorem 2. Any analytic solution to the system (22)–(25) and (29)–(33) is axially symmetric.

Proof. Let us consider a point $p \in H$, where

$$\partial_A q \neq 0, \qquad q \neq 0, \qquad q \neq F.$$

We call such points 'generic'. There is a whole neighbourhood, N say, of p, every point of which is generic. Then, q can be chosen as a coordinate in N, for example $x^2 = q$. We can also require in N that

$$g_{23} = 0 \tag{67}$$

and introduce the abbreviation

$$g_{22} = (g_2)^2, \qquad g_{33} = (g_3)^2, \tag{68}$$

where g_2 and g_3 are some positive functions in N. There remains the freedom

$$x^3 = f(x'^3)$$

with f an arbitrary function with non-vanishing gradient in N.

Choose the gauge with $B^2 = 0$. Then equation (47) implies that $B^3 \neq 0$ in N. Using (67) and (68) in (47), we obtain

$$U_2^3 = 0, \qquad U_3^3 \neq 0. \tag{69}$$

Equation (25) then becomes

$$U_2^2 = \tfrac{1}{2}(g_2)^2(U^2 - (g_2 q)^{-2}), \tag{70}$$

$$U_2^2 U_3^2 = 0, \tag{71}$$

$$(U_3^2)^2 + (U_3^3)^2 = \tfrac{1}{2}(g_3)^2(U^2 + (g_2 q)^{-2}). \tag{72}$$

Equation (71) admits two solutions

$$U_2^2 = 0 \qquad \text{in } N \tag{73}$$

or

$$U_3^2 = 0 \qquad \text{in } N. \tag{74}$$

We show that the solution (73) leads to a contradiction with the rest of the equations.

From (73) and (69), we have

$$U_2^a = 0.$$

This is a gauge-invariant statement (see (21)); in particular, the following equation for a gauge- and coordinate-invariant quantity follows from it:

$$(1/\sqrt{g})(U_2^2 U_3^3 - U_2^3 U_3^2) = 0, \tag{75}$$

and holds in the whole of N.

It is convenient to choose another gauge

$$U_2^a = 0, \qquad U_3^2 = 0. \tag{76}$$

Then (70) and (72) imply

$$U_3^3 = \pm g_3/g_2 q. \tag{77}$$

In the gauge (76), equations (30)–(32) read

$$B^1 = (g_2 g_3)^{-1}(\partial_2 W_3 - \partial_3 W_2), \tag{78}$$

$$B^2 = -(g_2 g_3)^{-1} W_2 U_3^3, \tag{79}$$

$$B^3 = (g_2 g_3)^{-1} \partial_2 U_3^3, \tag{80}$$

whereas equations (27)–(29) become

$$\partial_A B^1 - U_A^3 B^2 = 0, \tag{81}$$

$$\partial_A B^2 + U_A^3 B^1 = W_A B^3, \tag{82}$$

$$\partial_A B^3 + W_A B^2 = \delta_A^2 e^2 q^2 (g_2/g_3) U_3^3. \tag{83}$$

From (81), (79) and (76), we obtain

$$\partial_2 B^1 = 0, \tag{84}$$

$$\partial_3 B^1 = -(U_3^3)^2 W_2/g_2 g_3, \tag{85}$$

and (82) for $A=2$, (76), (79) and (80) yield

$$\partial_2[(U_3^3)^2 W_2/g_2 g_3] = 0. \tag{86}$$

(86) is the integrability condition for the system (84) and (85). B^1 is a gauge invariant quantity and so are the left-hand sides of (84), (85). In order that (85) holds, the gauge (76) must be chosen in N.

Next, consider a point p, where the function q attains its minimal value on H. Such a point must exist, as q is continuous and H is compact. Two cases are possible:

(a) There is a curve, Γ say, through p such that q is constant along Γ.

(b) There is a neighbourhood, N_1 say, of p in H such that $(\partial_A q)_r \neq 0$ and $q(r) > q(p)$ for any $r \in N_1$, $r \neq p$.

Assume (a). Then equation (25) implies at Γ

$$q^2 U_A^a U_B^a = \tfrac{1}{2} q^2 U^2 g_{AB}.$$

This leads either to

$$q(p) = 0$$

or to

$$U_2^a U_2^a = \tfrac{1}{2} U^2 (g_2)^2, \qquad U_3^a U_3^a = \tfrac{1}{2} U^2 (g_3)^2, \qquad U_2^a U_3^a = 0,$$

in any coordinates about Γ which satisfy (67) and (68). The last three equations mean either that

$$U^2 = 0$$

at Γ or that U_2^a and U_3^a are orthogonal linearly independent vectors in the internal space and so

$$(1/\sqrt{g})(U_2^2 U_3^3 - U_2^3 U_3^2) \neq 0$$

at Γ. However, every point of a neighbourhood of Γ, except for points at Γ, is generic, so (75) holds at it. From continuity, equation (75) must also be true at Γ, and we have either

$$q = 0 \qquad \text{or} \qquad U^2 = 0$$

at Γ. Now, look at equation (33). For $U^2 = 0$, it implies

$$(1/\sqrt{g})\partial_A(\sqrt{g} g^{AB} \partial_B q) + \tfrac{1}{2} k (F^2 - q^2) q = 0$$

along Γ. As q has a minimum at Γ, the first term, being the Laplace operator, is non-negative, and so is the second term (as $0 \leq q \leq F$). The equation can therefore be satisfied only with

$$(\partial_A \partial_B q)_\Gamma = 0 \qquad \text{and} \qquad (q)_\Gamma = 0,$$

because the second root of $(F^2 - q^2)q$, F, cannot correspond to a minimum. Hence, in any case

$$q|_\Gamma = 0, \qquad (\partial_A q)_\Gamma = 0, \qquad (\partial_A \partial_B q)_\Gamma = 0. \tag{87}$$

Using the analyticity of the function q, the existence and uniqueness theorem of Cauchy and Kovalevska, and the Cauchy data (87) along Γ, we conclude that (33) implies for q that

$$q = 0$$

in a whole neighbourhood of Γ. This is a contradiction for case (a).

Consider case (b). It is clear that all q-level-orthogonal curves in the neighbourhood N_1 of p meet at p. Along any such curve, $B^1 = \text{constant}$, because of (84). B^1, as a gauge- and coordinate-independent quantity, must be continuous at p. Hence

$$B^1 = c = \text{constant}$$

in the whole of N_1. Then (85) yields

$$W_2 = 0$$

at any generic point of N_1: there is a neighbourhood, N_2 say, of such a generic point, in which we can choose the gauge (76). In N_2, the magnetic equations (78)–(83) become

$$B^1 = (g_2g_3)^{-1}\partial_2 W_3 = c, \tag{88}$$

$$B^2 = 0, \tag{89}$$

$$B^3 = \pm(g_2g_3)^{-1}\partial_2(g_3/g_2q), \tag{90}$$

$$\partial_2 B^3 = \pm eq, \qquad \partial_3 B^3 = 0, \tag{91}$$

$$U_3^3 B^1 - W_3 B^3 = 0. \tag{92}$$

(We have used (77) where convenient.) Equation (91) can be integrated immediately,

$$B^3 = \pm\tfrac{1}{2}eq^2 + C,$$

where C is constant in N_2. Then (90) leads to

$$\partial_2(g_3/g_2q) = (\tfrac{1}{2}eq^2 \pm C)g_2g_3. \tag{93}$$

Equations (92), (88), (90) and (77) imply

$$\partial_2(g_3/g_2qW_3) = 0$$

or

$$W_3 = w^{-1}(x^3)g_3/g_2q.$$

Using (88) again, we obtain

$$\partial_2(g_3/g_2q) = cw(x^3)g_2g_3.$$

However, this is compatible with (93) only if $q = 0$ in N_2. Hence, case (b) is excluded, too, and we are left with equation (74). In the gauge where $B^2 = 0$, we obtain in N

$$U_A^2 = (u_2, 0), \qquad U_A^3 = (0, u_3),$$

where, from (70) and (72),

$$(u_2)^2 = \tfrac{1}{2}(U^2 - (g_2q)^{-2})(g^2)^2, \qquad (u_3)^2 = \tfrac{1}{2}(U^2 + (g_2q)^{-2})(g_3)^2. \tag{94}$$

The equations (27)–(32) become

$$\partial_3 B^1 = \partial_3 B^3 = \partial_3 u_2 = W_2 = 0, \tag{95}$$

$$\partial_2 B^1 + u_2 B^3 = 0, \qquad \partial_2 B^3 - u_2 B^1 = e^2 q^2 (g_2/g_3) u_3, \tag{96}$$

$$u_3 B^1 - W_3 B^3 = -e^2 q^2 (g_3/g_2) u_2, \tag{97}$$

$$B^1 = (g_2 g_3)^{-1} (\partial_2 W_3 + u_2 u_3), \tag{98}$$

$$B^3 = (g_2 g_3)^{-1} (\partial_2 u_3 - W_3 u_2). \tag{99}$$

Equations (33) and (34) read

$$(g_2 g_3)^{-1} \partial_2 (g_3/g_2) - qU^2 + \tfrac{1}{2} k (F^2 - q^2) q = 0, \tag{100}$$

$$\frac{q}{g_2 g_3} \partial_2 \left(\frac{g_3 u_2}{g_2} \right) + \frac{2u_2}{(g_2)^2} - \frac{q}{(g_3)^2} W_3 u_3 = 0, \tag{101}$$

$$\partial_2 (g_2 u_3 / g_3) = 0. \tag{102}$$

Multiplying (97) by g_2/g_3 and using (102) and (95), we have

$$\partial_3 (g_2 W_3 / g_3) = 0. \tag{103}$$

Similarly multiplying equations (98) and (99) by g_2^2 and using (102) and (103), we derive the following relations:

$$B^1 (\partial_3 g_2^2) = \frac{g_2 W_3}{g_3} \left(\partial_2 \partial_3 \log \frac{g_3}{g_2} \right), \tag{104}$$

$$B^3 (\partial_3 g_2^2) = \frac{g_2 u_3}{g_3} \left(\partial_2 \partial_3 \log \frac{g_3}{g_2} \right). \tag{105}$$

Suppose that $\partial_3 g_2^2 \neq 0$. Then (104) and (105) yield

$$B^1 u_3 - B^3 W_3 = 0.$$

Comparing this with (97), we conclude that $u_2 = 0$. However, this is precisely equation (73) which has led to a contradiction. Thus

$$\partial_3 g_2 = 0. \tag{106}$$

Now, (102) and (103) imply immediately

$$\partial_3 (u_3/g_3) = \partial_3 (W_3/g_3) = 0. \tag{107}$$

(98) and (99) are equivalent to the equations

$$\partial_2 u_3 / g_3 = B^3 g_2 + u_2 W_3 / g_3, \qquad \partial_2 W_3 / g_3 = B^1 g_2 - u_2 u_3 / g_3,$$

whose right-hand sides are independent of x^3. Hence, using (107), we obtain

$$\partial_3 \partial_2 \log u_3 = \partial_3 \partial_2 \log W_3 = 0$$

and so

$$u_3 = \bar{u}_3(q) \lambda(x^3), \qquad W_3 = \bar{W}_3(q) \mu(x^3).$$

Equations (107) then yield

$$g_3 = \bar{g}_3(q) \lambda(x^3) = \bar{\bar{g}}_3(q) \mu(x^3),$$

which is possible only if

$$\lambda(x^3) = K \mu(x^3)$$

for some constant K. Hence, we can write

$$u_3 = \bar{u}_3(q)\lambda(x^3), \qquad W_3 = \bar{W}_3(q)\lambda(x^3), \qquad g_3 = \bar{g}_3(q)\lambda(x^3).$$

If we perform the transformation of the coordinate x^3

$$x'^3 = \int \lambda(x^3)\, dx^3,$$

we can transform all the functions u_3, W_3 and g_3 to an x'^3-independent form:

$$\partial'_3 g'_3 = \partial'_3 u'_3 = \partial'_3 W'_3.$$

Summarising, we can say that there is a gauge and a coordinate system in N such that all independent variables of the theory become independent of one of the coordinates—namely x'^3. Hence, from analyticity, the solution admits a one-dimensional symmetry group. This group is, in particular, an isometry group on a two-dimensional Riemannian manifold with a compact topology. Thus, it is an 'axial symmetry'.

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